#### **2. Evolution of a biological species. Differentiation and differential equation**

**2.1. Introduction**

**Object** Evolution of one biological species

**Foundation** Differentiation operation and differential equations

**Aim** Definition of the law of change in the number of species depending on the conditions of the process

The function *x=x*(*t*) describes the number of species at the time *t.*

**Conclusion** We work with function change.

 Characteristic of the function change is the function derivative.

**2.2. Function derivative**

There exists three definition of the function derivative.

**Analytic definition**. *We have the function x=x*(*t*). *Its* ***derivative*** *at the point x is the limit x*'(*t*) *of the ratio of the function increament* [*x*(*t+τ*)–*x*(*t*)] *to the argument increament* *τ as τ tends to zero.*

**Geometric definition**. *We have the curve x=x*(*t*). *The ratio* [*x*(*t+τ*)–*x*(*t*)]/*τ is the tangent tangent of the secant angle. The* ***derivative*** *at the point x is the tangent of the correspondent tangent line.*



Figure 1. Geometric derivative definition

**Physical definition**. *The body is at the point x*(*t*) *at the time t*, *and at the point* *x*(*t+τ*) *at the time t+τ. Its average velocity is the ratio* [*x*(*t+τ*)–*x*(*t*)]/*τ. The corresponding limit as τ tends to zero, which is the* ***derivative****, is the velocity of the body at the time t.*

The velocity of the body describes how quickly the body moves.

The derivative of the function describes how quickly the function changes.

**Example 1**. Consider two functions: *x*(*t*)=*t* and *y*(*t*)=2*t.* Determine the ratio

[*x*(*t+τ*)–*x*(*t*)]/*τ* = 1, [*y*(*t+τ*)–*y*(*t*)]/*τ* = 2.

Then we have *x*'(*t*)=1, *y*'(*t*)=2. Therefore, the function *y* changes twice as fast as the function *x*.

**2.3. Malthus equation**

We first consider the simplest biological system represented by one species. The studied processes characterized by an ***numbers*** of the species *x*, which varies with time. Obviously, the velocity of change in the number of species is determined by the difference between its birthrate and mortality. Thus, we obtain the formula

*x*'(*t*) = *A – B*,

where the values of *A* and *B* characterize, respectively, the number of births and deaths of individuals in a unit of time. Their values are apparently proportional to the abundance of the species. Indeed, the more individuals there are, the more they will be born and die in a fixed time interval. As a result, we obtain the relations *A=ax*, *B=bx*, where the positive constants *a* and *b* are process parameters and characterize the birthrate and mortality of the species.

Thus, we considered process described by the formula

 *x*'(*t*)= *kx*(*t*), (1)

where the coefficient *k*=*a*–*b* is called the ***growth*** in the abundance of species, determines the change in the number of species per unit time and can take both positive and negative values.

The formula (1) is an equation with respect to the unknown function *x=x*(*t*). This is called the ***differential equation***. More concrete, we have the ***Malthus*** ***equation***.

**2.4. Cauchy problem**

**Example 2**. Consider the differential equation *x*'(*t*)=0. Obviously, any constant is its solution.

**Example 3**. Consider now the differential equation *x*'(*t*)=1. Obviously, the function *x*(*t*)=*t+c* satisfies this equation for all constant *c.* This is called the ***general solution*** of the differential equation

**Conclusion**. The solution of the differential equation depends from the arbitrary constant. If we would like determine the unique solution (***partial solution***), then it is necessary to add some additional information. For the practical situation, this is the value of the unknown function at the initial time that is called the ***initial condition***. The differential equation with an initial condition is called the ***Caushy problem***.

**Example 4**. Consider now the differential equation *x*'(*t*)=1 with the initial condition *x*(0)=0. The general solution of the equation is *x*(*t*)=*t+c*. However, unique partial solution *x*(*t*)=*t* satisfies the given initial condition.

**2.5. Malthus model**

Return to the consideration of the evolution of a biological species. This is described by the Malthus equation (1), we the function *x=x*(*t*) is ther numbers of species at the concrete time. Suppose we know also the numbers of species at the initial time *t=*0, i.e., we have the initial condition

 *х*(0) = *х*0, (2)

where the initial number of species *х*0 is the parameter of the problem, taking any positive values. The Caushy problem (1), (2) has a unique solution. This is our first mathematical model of the system. This is called the ***Malthus model***.

In principle, for solving the problem (1), (2), it is necessary to find the general solution of the differentiational equation (1). This is depends froman arbitraly constant. It can be found using the initial condition (2). After that, we can put this constant to the formula of the general solution and find the solution of the Malthus model. However, we will not solve differential equations. For our analysis, it will be sufficient to use easiest properties of the derivatives.

**2.6. Easiest properties of the derivative**

Return to the definition of the function derivative. Consider the ratio [*x*(*t+τ*)–*x*(*t*)]/*τ*. This value is positive if the finction *x* increases and this is negative if this function decrease. Then the sign of the derivative determine the direction of the function change at the concrete point *t*. If the derivative is zero, then the velocity of the function change is zero, and the function does not change.

**2.7. Properties of the Malthus model**

Return to the Caushy problem (1), (2). By the equality (1) the properties of the function *x* on the sign of the parameter *k*.

If *k*>0, then the function *x* increases. Moreover, after increasing of the function *x*, its velocity of change *x*'=*kx* increases too. Therefore, we observe an exponential growth of the function *x*. The value of the function *x* tends to infinity.

If *k<*0, then the function *x* decreases. Moreover, after decreasing of the function *x*, its velocity of change *x*'=*kx* decreases too. Therefore, we observe a gradual decrease of the function *x*. The value of the function *x* tends to zero.

If *k=*0, the the function *x* has the zero velocity of change. Hence, *x*(*t*)=*x*0 for all time *t.*



Figure 2. Variants of evolution of the species number
with different ratios between birthrate and mortality.

Now we try to interpret the results. Negative values ​​of the coefficient *k* correspond to the unfortunate case when the mortality of a species prevails over its birthrate. Therefore, in a unit of time, the abundance of species decreases by a certain value. Next time, more individuals will die than be born. Since in this model the increase in the abundance of species is considered unchanged, we are observing its gradual extinction. This sad outcome is characteristic of a species found in extremely unfavorable living conditions caused by an acute shortage of food, a sharp deterioration of the habitat, the appearance of natural opponents, the spread of epidemics and other troubles.

Positive values ​​of the increase in the species abundance correspond to the excess of birthrate over mortality. This means that at any time interval more individuals are born than they die. Then, in the next interval, with the same increase, an even greater number of individuals will born. Thus, an unlimited increase in the number of species is observed, which is characteristic of a species under ideal conditions, i.e. in the absence of opponents and competitors, with an unlimited supply of food and a favorable environment. A similar situation is possible, for example, for microorganisms in a nutrient medium.

The trivial case of zero increase in numbers means that birthrate and mortality are balanced. The number of dead individuals here is compensated by the newly born, and the species abundance remains unchanged.

**2.8. Verhulst model**

The described model has a fairly obvious flaw. With a positive increase in the species, an exponential increase in its abundanceis observed here, which is in poor agreement with reality. In nature, the increase in the species abundanceis constrained by restrictions on the amount of available food and free territory, the presence of natural opponents and competitors, etc. We restrict ourselves to considering the evolution of the species in conditions of limited food intake. Then, the increase in the species abundancein relation (7.1) will already substantially depend on the function *x*. It can be assumed that this dependence has the following form

*k*(*х*)*=а*(*D – qx*) – *b*,

where *D* is the amount of incoming food, *q* is the coefficient characterizing the consumption of food, *b* is the natural mortality of the species (not related to lack of food), and is the specific increase in the number of species (growth corresponding to the unit of incoming food). Under the assumptions made, the birthrate of the species is directly proportional to the excess of the amount of incoming food over the amount of necessary food (with a proportionality coefficient *a*). In case of food shortage, a corresponding increase in the mortality rate of the species is observed.

 *x*'(*t*) *=* [(*аD*–*b*)–*аqx*(*t*)] *x*(*t*). (3)

This is called the ***Verhulst equation*** or the ***logistic equation***. After adding the initial condition (2), we get the Caushy problem (2), (3) that is called the ***Verhulst model***. For its analysis, it is necessary to give additional information about differential equations.

**2.9. Equillibrium state**

Consider a differential equation

 *x*' *= f*(*x*), (4)

where *f* is a given function. Suppose a number *x* satisfies the algebraic equation

 *f*(*x*) = 0. (5)

If we chose value as an initial condition for the equality (4), then the derivative of the function *x* at the initial time is zero. However, the derivative of the function is the velocity of its change. Therefore, the change of this function is equal to zero at the initial time, so this function does not change. The solution of the equation (5) is called the ***equilibrium state*** for the differential equation (4).

**Conclusion**. If we chose an equilibrium state as an initial state, then the system is in this state, and we have no evolution.

Consider the Malthus equation. The corresponding equality (5) is *x*=0. Hence, the equilibrium state for the Malthus equation is zero. Obviously, if the number of species at the initial time is zero, then we have no species. Therefore, the number of species will be zero for any time.

The question arise, what we obtain, if the initial state system is close enough to the equilibrium state, i.e., in its small neighborhood. For the Malthus equation, we can have two different situation; see Figure 2. If *k*<0, then the function *x* tends to the equilibrium state; and for *k*>0, it does not tend. These cases are typical for the general equation (4). If for all *x*0 from a small neighborhood the function *x=x*(*t*) tends to the equilibrium state, then this is the ***stable equilibrium state***, else it is ***non-stable***.

**Conclusion**. If the growth coefficient *k* for the Malthus equation is negative, then the trivial state of the system is stable. If this coefficient is positive, then the equilibrium state is non-stable.

The physical interpretation of the stable equilibrium state is the position of the ball in the pit. However, the position of the ball on the mountain is non-stable; see Figure 3.

Now we can return to the analysis of the Verhulst equation.

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Figure 3. Stability and non-stability of equilibrium positions.

**2.10. Analysis of the Verhulst equation**

Consider the Verhulst equation (3). The corresponding equality (5) is

 [(*аD*–*b*)–*аqx*] *x =* 0. (6)

We have the square equation. It has two solutions. First of them is trivial *x*=0, and the second solution is

*x*\* = (*D–b*/*a*)*q.*

Hence, the Verhulst equation has two equilibrium positions. Try to analyze both. At first, we determine a mathematical result. After that, we will try to give a biological interpretation.

The sign of the derivative of the function *x* is determined by the term in square brackets on the right-hand side of the formula (6). Under the inequality *аD*≤*b*, this derivative is obviously negative. Thus, the function *x* monotonically decreases. However, as it approaches zero, its derivative steadily tends to zero. Thus, the rate of decrease of the state function gradually slows down, and its value tends to zero. In this situation, the trivial equilibrium state is stable, and non-trivial one is not stable.

For *аD***>***b*, the sign of the derivative depends on the current state of the system. If the initial number *х*0 is less than the value of *х*\*, then the derivative *x*'at the initial time is positive, which means that the function *x* increases. This situation is observed all the time, while the inequality *х***<***х*\* is true. However, as *x* approaches the critical value of *х*\*, according to equality (3), the derivative *x*'tends to zero. This suggests that as *t* grows, the value of *x*(*t*), increasing, tends to *х*\*. Therefore, the non-trivial equilibrium state is stable, and the trivial on is non-stable, see Figure 4.



Figure 4. Species abundance tends to *х*\* if the quantity of food is bounded.

For *x*0 **>***x*\*, the derivative *x*'at the initial time is negative, and therefore, the function *x* decreases. As it decreases, the value of *x* approaches *х***\***, and the derivative *x*'tends to zero. Thus, at *t*→∞, the function *x* monotonically decreases and tends to *х*\*. Finally, for *x*0**=***x*\* the derivative ** is equal to zero, which means that the quantity *x* does not change, remaining equal to *х*\*. Therefore, regardless of the initial state of the system, the solution of equation (3) tends to the value *х*\*, see Figure 4.

The obtained results have a natural interpretation. If *аD*≤*b*, the the mortality is greater then the birthrate, and the population die out. Besides, this result does depend from the initial number of species. For *аD***>***b*, we have another result. If the initial number of a species is small enough, then the boundedness on food supplies is not so important, and the number of individuals is growing steadily. However, at the same time, the quantity of food consumed (with its constant supply) will certainly increase. A steadily increasing shortage of food restrains the growth of the species abundance. As a result, a certain equilibrium value of the number of species *х*\* is gradually established, which can be maintained at a given ratio between the amount of food available and its consumption. If initially the abundance of the species is too large, then there is a shortage of food, which leads to a reduction in the abundance of the species. However, as the abundance of the species decreases, food consumption decreases. Thus, the lack of food is gradually reduced, which means that the role of negative factors affecting the behavior of the system is gradually disappearing. One way or another, over time, one and the same equilibrium value *х*\* of the number of species is established. Variants of evolution of the considered system are given in Figure 5.



Figure 5.Variants of the system evolution for the Verhulst model.

Note that the factors restraining an increase in the species abundance can also be the presence of a competing species consuming the same food, or natural opponents for which this species is itself food. Thus, we come to the study of the coexistence of several biological species. In this case, the decisive role is played by the principle of interaction between species.